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Mean Quadratic Variations and Fourier Asymptotics of Self-similar Measures

By

Ka-Sing Lau and **Jianrong Wang**, Pittsburgh

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Abstract. We show that the mean quadratic variation of a self-similar measure μ under certain open set condition exhibits asymptotic periodicity. Through a generalized Wiener's Tauberian Theorem, we obtain some new identities and equivalences of the mean quadratic variation of a bounded measure ν and its Fourier average $H_\alpha(T; \nu) = \frac{1}{T^{n-\alpha}} \int_{|x| \leqslant T} |\hat{\nu}(x)|^2 dx$ $(0 \leqslant \alpha \leqslant n)$. They are used to sharpen some recent results of Strichartz concerning the asymptotic behavior of $H_\alpha(T; \mu)$ as $T \to \infty$, where μ is the self-similar measure as above. In the development some results concerning the open set condition are also obtained.

1. Introduction

Let μ be a σ-finite Borel measure on $\mathbf{R}^n$, for $0 \leqslant \alpha \leqslant n$, let

$$V_\alpha(t; \mu) = \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |\mu(B_t(x))|^2 dx$$

where $B_t(x)$ is the ball of radius t, centered at x. We will call $\limsup_{t \to 0} V_\alpha(t; \mu)$ the *upper α-mean quadratic variation* (m.q.v.) of μ, and simply call it α-m.q.v. if the limit exists. The *m.q.v. index* of μ is defined to be

$$\inf\{\alpha\colon \limsup_{t \to 0} V_\alpha(t; \mu) > 0\},$$

which obviously equals to $\sup\{\alpha\colon \limsup_{t \to 0} V_\alpha(t; \mu) < \infty\}$. If μ contains an atom, then the 0-m.q.v is $\sum |\mu\{x\}|^2$ and the index is hence 0. On the other hand, if μ is absolutely continuous with density function h in $L^2(\mathbf{R}^n)$, then, by observing that

$$\lim_{t \to 0} \frac{1}{(2t)^{2n}} \int_{\mathbf{R}^n} |\mu(Q_t(x)|^2 dx = \int_{\mathbf{R}^n} |h(x)|^2 dx$$

(see [HL] for $n = 1$), where $Q_t(x)$ is the cube centered at x with length $2t$ on each side, μ has m.q.v. index n. In some sense the m.q.v. index measures the degree of singularity of a measure μ on $\mathbf{R}^n$.

Let f be a Borel measurable function on $\mathbf{R}^n$, we use μ_f to denote the measure defined by $\mu_f(E) = \int_E f\, d\mu$ for any Borel set E in $\mathbf{R}^n$. Recently, STRICHARTZ [S1] observed that several classical identities can be summarized in the following form where $0 \leqslant \alpha \leqslant n$ are integers:

$$\lim_{T\to\infty} \frac{1}{T^{n-\alpha}} \int_{|x|\leqslant T} |(\mu_f)^{\widehat{}}(x)|^2\, dx = \int |f|^2\, d\mu \quad \text{for any } f \in L^2(d\mu). \quad (1.1)$$

E.g. the Plancherel identity ($\alpha = n$, μ is the Lebesgue measure); the Wiener identity for bounded measures, and the Besicovitch identity for almost periodic functions ($\alpha = 0$); the identity of Agmon and Hörmander ($\alpha = 0, 1, 2, \ldots, n-1$, μ is the surface measure on a C^1-submanifold). He then introduced the concept of locally uniformly α-dimensional measure and proved among many interesting results that for such measures

$$\limsup_{T\to\infty} \frac{1}{T^{n-\alpha}} \int_{|x|\leqslant T} |(\mu_f)^{\widehat{}}(x)|^2\, dx \leqslant C \int |f|^2\, d\mu \qquad \text{for any } f \in L^2(d\mu). \tag{1.2}$$

By using the m.q.v. (actually the mean p-variation) (1.2) has been extended to the p, q form, and a necessary and sufficient condition of μ for such inequality has also been obtained [L1].

Such inequalities (or identities) can be approached by the mean quadratic variation through the following equivalence (Corollary 4.5):

$$\limsup_{T\to\infty} \frac{1}{T^{n-\alpha}} \int_{|x|\leqslant T} |(\mu_f)^{\widehat{}}(x)|^2\, dx \approx \limsup_{t\to 0} \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |\mu_f(B_t(x))|^2\, dx,$$

as is used in [L1]. The relationship of the two expressions has a long history. It dates back to WIENER's generalized harmonic analysis and his Tauberian Theorem (We call the identity there Wiener-Plancherel formula [Wi], where $\alpha = 0$, $\hat{\mu}_f$ and $\mu_f(B_t(x))$ are replaced by more general functions, and only the limit case is considered).

Following the notations of Hutchinson [H], we let $\{S_j\}_{j=1}^m$ be contractive similarities on $\mathbf{R}^n$, i.e. $S_j(x) = \varrho_j R_j x + b_j$, $j = 1, \ldots, m$, $x \in \mathbf{R}^n$, where $0 < \varrho_j < 1$, R_j are orthogonal transformations and $b_j \in \mathbf{R}^n$. For any positive weights $\{a_j\}_{j=1}^m$ there is a unique Borel probability measure μ with compact support such that

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}.$$

we will call such μ a self-similar measure (with respect to $\{S_j\}_{j=1}^m$ and $\{a_j\}_{j=1}^m$).

The similarities $\{S_j\}_{j=1}^m$ are said to satisfy the open set condition if there is a bounded open set U such that

$$S_j U \subseteq U \quad \text{for all} \quad j, \quad \text{and} \quad \{S_j U\}_{j=1}^m \quad \text{are disjoint.}$$

Note that $\operatorname{supp}(\mu) \subseteq \overline{U}$.

In **[S2]**, **[S3]** STRICHARTZ investigated the Fourier asymptotic behavior of self-similar measures. He showed that if $\{S_j\}_{j=1}^m$ satisfies the 'strong' open set condition ($\{S_j(\overline{U})\}_{j=1}^m$ are disjoint), then the asymptotic behavior of $\hat{\mu}$ is given by

$$\frac{1}{T^{n-\beta}} \int_{|x| \leq T} |\hat{\mu}(x)|^2 \, dx = q(T) + E(T), \tag{1.3}$$

where β satisfies $\sum_{j=1}^m a_j^2 \varrho_j^{-\beta} = 1$, and either (i) $q(T) \equiv c > 0$ and $\lim_{T \to \infty} E(T) = 0$; or (ii) $q(T) > 0$ is a multiplicative periodic function and $\lim_{T \to \infty} \int_T^{\lambda T} |E(r)| \, dr/r = 0$, according to $\{-\ln \varrho_j \colon j = 1, \ldots, m\}$ is non-arithmetic or arithmetic.

It is also proved in **[S3]** that if $\{S_j\}_{j=1}^m$ satisfies the strong open set condition, then for the self-similar measure μ defined by the natural weights (i.e. $a_j = \varrho_j^{-\beta}$, $j = 1, \ldots, m$),

$$\frac{1}{T^{n-\beta}} \int_{|x| \leq T} |(\mu_f)\hat{\ }(x)|^2 \, dx = q(T) \int |f|^2 \, d\mu + E(T) \quad \text{for any } f \in L^2(d\mu), \tag{1.4}$$

where $E(T) \to 0$ in the above sense.

In this paper we extend the above results through the mean quadratic variation.

Theorem A. *Suppose that the similarities $\{S_j\}_{j=1}^m$ satisfy the open set condition with respect to an open set U, and μ is a self-similar measure such that $\mu(\partial U) = 0$, then for β satisfying $\sum_{j=1}^m a_j^2 \varrho_j^{-\beta} = 1$,*

$$\lim_{t \to 0} \left[\frac{1}{t^{n+\beta}} \int_{\mathbf{R}^n} |\mu(B_t(x))|^2 \, dx - P(t) \right] = 0$$

where either (i) $P \equiv c > 0$, *if* $\{-\ln \varrho_j : j = 1, \ldots, m\}$ *are non-arithmetic, or* (ii) $P > 0$ *is a multiplicative periodic function with period* ϱ *where* $(\ln \varrho)\mathbf{Z}$ *is the lattice generated by* $\{-\ln \varrho_j : j = 1, \ldots, m\}$.

Theorem B. *If in addition the weights are natural, then*

$$\lim_{t \to 0} \left[\frac{1}{t^{n+\beta}} \int_{\mathbf{R}^n} |\mu_f(B_t(x))|^2 \, dx - P(t) \int |f|^2 \, d\mu \right] = 0 \quad \textit{for all } f \in L^2(d\mu).$$

We do not know whether the condition $\mu(\partial U) = 0$ is necessary, however it is much weaker than the strong open set condition. It is also true that for most well known cases (e.g. Koch curve, Sierpinski triangle, etc), the condition is satisfied while the strong open condition fails. Furthermore we have the following dichotomous result.

Theorem C. *Let* U *be the open set defined in the open set condition, and let* μ *be a self-similar measure, then either* $\mu(\partial U) = 0$ *or* $\mu(\partial U) = 1$.

We remark that the m.q.v. indices are much harder to obtain for the self-similar measures that do *not* satisfy the open set condition. Some special cases are considered in **[L1]** and **[L2]**.

In order to transfer the above results to the Fourier asymptotic averages (hence (1.3) and (1.4) will follow as corollaries and are sharpened), we establish the following extended form of Tauberian Theorem which covers, in particular, the periodic case.

Theorem D. *The following conditions are equivalent*:

$$\lim_{t \to 0} \left[\frac{1}{t^{n+\beta}} \int_{\mathbf{R}^n} |W_t(g)(x)|^2 \, dx - P(t) \right] = 0,$$

$$\lim_{T \to \infty} \left[\frac{1}{T^{n-\beta}} \int_{|y| \leqslant T} |g(y)|^2 \, dy - Q(T) \right] = 0;$$

where P *and* Q *are bounded multiplicative periodic with same period.*

Here W_t is an analog of the Wiener transformation on $\mathbf{R}$ **[Wi]**, and is acting on $B_\alpha^2(\mathbf{R}^n)$: the class of g with bounded α-mean quadratic averages. As a special case we take $g = \hat{\mu}$, then $W_t(g)(x) = \mu(B_t(x))$ for almost all x (Proposition 4.3). We remark that several other versions of n-dimensional Wiener transformation and Wiener-Plancherel formulas have been obtained in **[B]**, **[BBE]**, **[Be]**, **[He]**.

We organize the paper as following: In Section 2, we discuss the open set condition and prove Theorem C (Theorem 2.3). Some other basic properties and relevant results are also presented. Theorem A (Theorem 3.7) is proved in Section 3, the main idea is to use the self-similar properties of μ to reduce the mean quadratic variation to a renewal equation. The solution of such equation is well known **[F]**, and turns out to be the cases (i) and (ii) of Theorem A (The renewal equation has been used in **[La]** for calculating the packing dimension). In Section 4, we define the extended Wiener transformation $W_t(f)$ (4.1) for functions f in $B^2_\alpha(\mathbf{R}^n)$, and prove Theorem D (Theorem 4.10) in a more general setting. Our approach follows from ([**T**], Charpter XII, § 5). Several extensions of Wiener-Plancherel formula are obtained. Finally in Section 5, we sum up the previous results so that the statements for the Fourier asymptotics of self-similar measures follow readily (Theorem 5.1 and Theorem 5.3).

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2. Open Set Condition

For any fixed $m \in \mathbf{N}$, we use $J = (j_1, \ldots, j_k)$ to denote the multi-index, $|J| = k$ its length, and Λ the set of all such multi-indices, where $j_i \in \{1, \ldots, m\}$, $i = 1, \ldots, k$ and $k \in \mathbf{N}$. For any $\{c_i\} \in \mathbf{R}$ and for any m maps S_i: $\mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \ldots, m$, we set

$$c_J = c_{j_1} \ldots c_{j_k}, \qquad S_J = S_{j_1} \circ \ldots \circ S_{j_k}.$$

Also for any set $A \in R^n$ we set $A_J := S_J(A)$ (also $A_j = S_j(A)$) and use these two notations interchangably.

Let $\{S_j\}_{j=1}^m$ be contractive similarities, i.e. $S_j x = \varrho_j R_j x + b_j$, $x \in \mathbf{R}^n$, where $0 < \varrho_j < 1$, R_j are orthogonal transformations and $b_j \in \mathbf{R}^n$, $j = 1, \ldots, m$. It is well known that there exists a compact subset E in $\mathbf{R}^n$ invariant under S_j, i.e. $E = \bigcup_{j=1}^m S_j(E)$ **[F]**, **[H]**. For any $a_j > 0$, $j = 1, \ldots, m$ with $\sum_{j=1}^m a_j = 1$, there exists a unique Borel probability measure μ with support contained in E such that

$$\mu = \sum_{j=1}^{m} a_j \mu_j \tag{2.1}$$

where $\mu_j(F) = \mu(S_j^{-1}(F))$ for any Borel subset F in $\mathbf{R}^n$. A measure, which satisfies (2.1), is called a *self-similar measure*. It follows easily that

$$\mu = \sum_{|J|=k} a_J \mu_J$$

where $\mu_J(F) = \mu(S_J^{-1}(F))$.

The contractive similarities $\{S_j\}_{j=1}^m$ is said to satisfy *the open set condition* if there exists a bounded open set U such that

$$S_i(U) \subseteq U \quad \text{and} \quad S_i(U) \cap S_j(U) = \phi \tag{2.2}$$

for all $i \neq j \in \{1, \ldots, m\}$. It is proved in **[H]** that the invariant set E is contained in $\bar{U}$ and $\bigcup_{|J|=k} S_J(\bar{U})$ for each k; if s is such that $\sum_{i=1}^m \varrho_i^s = 1$, then $0 < \omega_s(E) < \infty$, where ω_s is the s-dimension Hausdorff measure; if we take $a_j = \varrho_j^s$ (the natural weight), then the self-similar measure μ defined by (2.1) equals $c\,\omega_s|_E$ where $c = \omega_s(E)^{-1}$ ($\omega_s|_E$ means ω_s restricted on E), hence the measure μ satisfies

$$\mu(E_j) = \varrho_j^s \mu(E) = a_j, \quad \text{and} \quad \mu(E_i \cap E_j) = 0 \quad \text{for} \quad i \neq j. \tag{2.3}$$

We do not know whether this holds for arbitrary positive weight $\{a_i\}_{i=1}^m$ such that $\sum_{i=1}^m a_i = 1$, however we have

Proposition 2.1. *Let* μ *be the invariant measure defined by* (2.1) *and* U *satisfy the open set condition with* $\mu(\partial U_i \cap \partial U_j) = 0$ *(equivalently,* $\mu(\bar{U}_i \cap \bar{U}_j) = 0$) *for* $i \neq j$, *then* $\mu(\bar{U}_i) = a_i$.

Proof. Since

$$a_i \mu(S_i^{-1}(\bar{U}_i \cap \bar{U}_j)) \leqslant \sum_k a_k \mu(S_k^{-1}(\bar{U}_i \cap \bar{U}_j)) = \mu(\bar{U}_i \cap \bar{U}_j) = 0,$$

it follows that

$$\mu(S_i^{-1}(\bar{U}_j)) = \mu(S_i^{-1}(\bar{U}_i \cap \bar{U}_j)) = 0 \quad \text{for} \quad i \neq j,$$

and hence for any $j = 1, \ldots, m$,

$$\mu(\bar{U}_j) = \sum_{i=1}^m a_i \mu(S_i^{-1}(\bar{U}_j)) = a_j \mu(S_j^{-1}(\bar{U}_j)) = a_j. \quad \text{Q.E.D.}$$

In the following, we will prove some dichotomous results for the self-similar measure μ on the open set U in (2.2). For any set F, we use

$\overline{F}$, ∂F to denote its closure and its boundary respectively. We begin with a set theoretic lemma.

Lemma 2.2. *Let U be a bounded open set satisfying* (2.2), *then for $i, j = 1, \ldots, m$:*

(i) $S^{-1}(U_i) \cap \overline{U} = U$.
(ii) $S^{-1}(U_j) \cap \overline{U} = \phi$ *for* $i \neq j$.
(iii) $S_i^{-1}(F) \cap \overline{U} \subseteq F$, $S_i^{-1}(U \backslash F) \cap \overline{U} \subseteq U \backslash F$ *where* $F = (\bigcup_{J \in \Lambda} \partial(U_J)) \cup (\partial U)$.
(iv) $\partial U \cap E \subseteq \bigcup_j (\partial U_j \cap E) \subseteq \bigcup_{J \in \Lambda} (\partial(U_J) \cap E)$.

Proof. (i) is trivial since $S_i^{-1}(U_i) = U$. To prove (ii) we observe that otherwise

$$S_i^{-1}(U_j) \cap \overline{U} \neq \phi \Rightarrow U_j \cap \overline{U}_i \neq \phi \Rightarrow U_j \cap U_i \neq \phi$$

which contradicts the open set condition.

For (iii) we let $x \in S_i^{-1}(F) \cap \overline{U}$, then $S_i(x) \in F$ so that $S_i(x) \in \partial U$ or $S_i(x) \in \partial U_J$ for some J. In the first case we observe that $x \notin U$ (otherwise $S_i(x) \in U_i \subseteq U$), hence $x \in \partial U \subseteq F$. In the second case write $U_J = S_{j_1} \circ \ldots \circ S_{j_k}(U)$. If $i = j_1$, then $x \in \partial(S_{j_2} \circ \ldots \circ S_{j_k}(U)) \subseteq F$; if $i \neq j_1$, then $S_i(x) \in \partial U_i$ (since $U_i \cap U_J \subseteq U_i \cap U_{j_1} = \phi$, and $S_i(x) \in \overline{U_i} \cap \overline{U_j}$) so that $x \in \partial U \subseteq F$. This completes the proof of the first inclusion. The second inclusion follows from the fact that $x \in F$ implies that $S_i x \in F$.

Finally, to prove (iv) we let $x \in \partial U \cap E$. Since $E \subseteq \bigcup_{j=1}^m \overline{U}_j$, $x \in \overline{U}_j$ for some j; that $x \in \partial U$ then implies that $x \in \partial U_j \cap E$. Q.E.D.

Theorem 2.3. *Let $\{S_j\}_{j=1}^m$ be contractive similarities and let U be an open set satisfying* (2.2). *Let μ be a self-similar measure defined by* (2.1), *then either*

$$\mu(U) = 1 \quad or \quad \mu(U) = 0.$$

Proof. Using Lemma 2.2 (i), (ii) we have

$$\mu(U_i) = \sum_j a_j \mu(S_j^{-1}(U_i)) = \sum_j a_j \mu(S_j^{-1}(U_i) \cap \overline{U}) = a_i \mu(U). \quad (2.4)$$

Inductively we have $\mu(U_J) = a_J \mu(U)$, hence for any $k \geqslant 1$,

$$\sum_{|J|=k} \mu(U_J) = \sum_{|J|=k} a_J \mu(U).$$

Also for any $k \geqslant 1$,

$$\mu(\partial U)+\mu(U)=\mu(\overline{U})=1=\mu\Big(\bigcup_{|J|=k}\overline{U}_J\Big)=$$

$$=\mu\Big(\bigcup_{|J|=k}U_J\Big)+\mu\Big(\bigcup_{|J|=k}\partial U_J\Big)=\mu(U)+\mu\Big(\bigcup_{|J|=k}\partial U_J\Big)$$

implies that

$$\mu(\partial U)=\mu\Big(\bigcup_{|J|=k}\partial U_J\Big). \tag{2.5}$$

Since $\operatorname{supp}\mu\subseteq E$, it follows from Lemma 2.2(iv) that

$$\mu\Big(\Big(\bigcup_{|J|=k}\partial U_J\Big)\setminus\partial U\Big)=0.$$

Let F be defined as in Lemma 2.2(iii), then the above implies that $\mu(F\setminus\partial U)=0$ and hence

$$\mu(U\setminus F)=\mu(U\setminus\partial U)=\mu(U).$$

If $\mu(U)\neq 0$ we define $\nu=\mu(U)^{-1}\mu|_{U\setminus F}$. Using Lemma 2.2 (iii) and the invariant property of μ, we can check that $\nu(A)=\sum_{i=1}^m a_i\nu_i(A)$ for Borel subsets $A\subseteq U\setminus F$, $A\subseteq F$, and $A\subseteq R^n\setminus\overline{U}$. Since ν is also a probability measure, the uniqueness of the invariant measure satisfying (2.1) implies that $\nu=\mu$. Hence $\mu(U)=1$ and the theorem is proved. Q.E.D.

The following corollaries follow directly from the theorem.

Corollary 2.4. *Under the hypotheses of Theorem 2.3, the measure μ is supported by either U or ∂U. If $S_J(\overline{U})\subseteq U$ for some $J\in\Lambda$, then $\mu(U)=1$ and $\mu(\partial U)=0$.*

Proof. If $\overline{U}_J=S_J(\overline{U})\subseteq U$ for some $J\in\Lambda$, then

$$\mu(U)\geqslant\mu(\overline{U}_J)=\sum_{|J'|=|J|}a_{J'}\mu\circ S_{J'}^{-1}(\overline{U}_J)\geqslant a_J\mu(\overline{U})=a_J>0. \quad \text{Q.E.D.}$$

Corollary 2.5. *Under the hypotheses of Theorem 2.3, the following are equivalent:*

(i) $\mu(U)=1$.

(ii) $\mu(U_i)=a_i$ *for some (and hence all)* $i=1,\ldots,m$.

(iii) $\mu(\partial U_i)=0$ *for some (and hence all)* $i=1,\ldots,m$.

Furthermore either one of the conditions above will imply (2.3).

Proof. (i)$\Leftrightarrow$(ii) follows from (2.4); (i)$\Rightarrow$(iii) follows from the theorem and (2.4). To prove (iii)$\Rightarrow$(i), let $\mu(\partial U_i) = 0$, for some i, then

$$\mu(U_i) = \mu(\overline{U}_i) = \sum_j a_j \mu_j(\overline{U}_i) \geqslant a_i \mu(\overline{U}) = a_i,$$

and hence by (2.4), $\mu(U) = 1$.

The last statement is clear since $E \subseteq \overline{U}$, $\mu(E_j \cap E_k) \leqslant \mu(\overline{U}_j \cap \overline{U}_k) = 0$. Q.E.D.

As a simple example for Theorem 2.3 we consider S_1, S_2 defined on $\mathbf{R}$ by

$$S_1(x) = \frac{1}{3}x, \qquad S_2(x) = \frac{1}{3}x + \frac{2}{3},$$

then the invariant set E is the Cantor set. If we let $U = (0, 1)$, and let μ be the Cantor measure ($a_1 = a_2 = \frac{1}{2}$), then $\mu(U) = 1$ of course. On the other hand if we take $V = (0, 1) \setminus E$, then V also satisfies (2.2) but $\mu(V) = 0$. This actually holds in a more general case.

Proposition 2.6. *If $\{S_j\}_{j=1}^m$ satisfies the open set condition, then the open set U can be chosen so that $\mu(U) = 0$ for any self-similar measure μ satisfying* (2.1).

Proof. Let V be an open set satisfies (2.2), and let $U = V \setminus E$. Then U is an open set with $\{S_i(U)\}_{i=1}^m$ disjoint and $\mu(U) = 0$. Also $S_i U \subseteq U$ for $i = 1, ..., m$ follows from

$$S_i(U) = (S_i V) \setminus (S_i E) = V_i \setminus (E \cap \overline{V}_i) = V_i \setminus E \subseteq V \setminus E = U. \quad \text{Q.E.D.}$$

Following the notation of **[S3]**, we say that $\{S_j\}_{j=1}^m$ satisfies *the strong open set condition* if there exists U such that $\{S_i(\overline{U})\}_{i=1}^m$ are disjoint and $S_i U \subseteq U$. Let

$$d(A, B) = \inf\{|x - y|: x \in A, y \in B\}$$

denote the distance of two sets A and B.

Proposition 2.7. *If $\{S_j\}_{j=1}^m$ satisfies the strong open set condition, then the open set U can be chosen so that $\mu(U) = 1$ for any self-similar measure μ satisfying* (2.1).

Proof. Let V satisfy the strong open set condition, let $\delta =$

$= \frac{1}{2} \min \{d(V_i, V_j): i \neq j,\ i, j = 1, \ldots, m\}$, the assumption implies that $\delta > 0$. Let $U = \{x: d(x, V) < \delta\} \supseteq V$, then

$$U_j \subseteq \{x: d(x, V_j) < \varrho_j \delta\} \subseteq \{x: d(x, V) < \delta\} = U$$

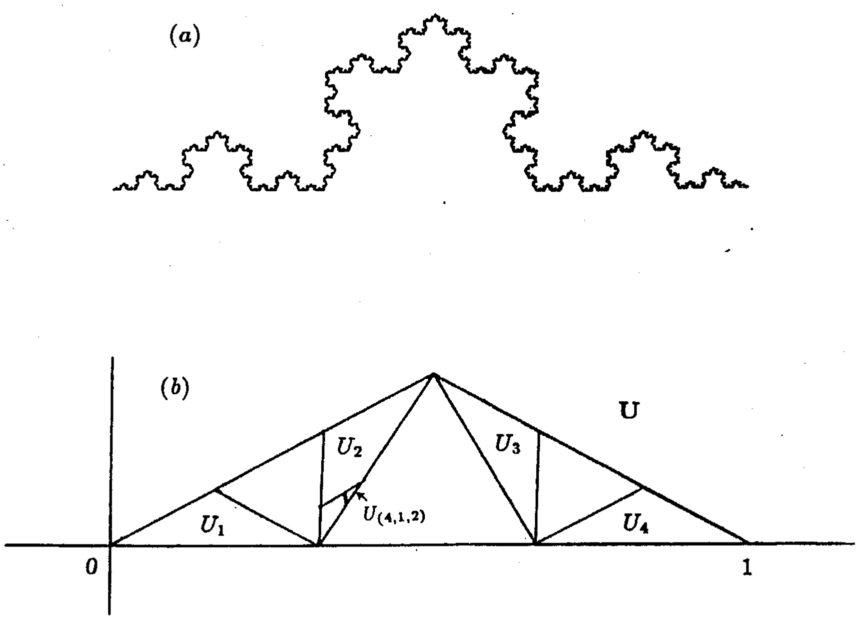
for any j, and clearly for $j \neq k$,

$$d(U_j, U_k) \geqslant d(V_j, V_k) - \varrho_j \delta - \varrho_k \delta > 2\delta - (\varrho_j + \varrho_k)\delta > 0.$$

Hence U satisfies the strong open set condition and that $E \subseteq \overline{V} \subseteq U$ implies $\mu(U) = 1$. Q.E.D.

In view of corollary 2.5 and all the standard examples, we see that $\mu(U) = 1$ is the natural case. We do not know whether in general such U can be constructed where $\{S_j\}_{j=1}^m$ satisfies the open set condition; nevertheless all of the known examples of self-similar measure μ do satisfy the sufficient condition $\overline{U_J} \subseteq U$ for some $J \in \Lambda$ so that $\mu(U) = 1$ (Corollary 2.4). On the other hand most of the examples do not satisfy the strong open set condition.

Example 1 (Koch curve).



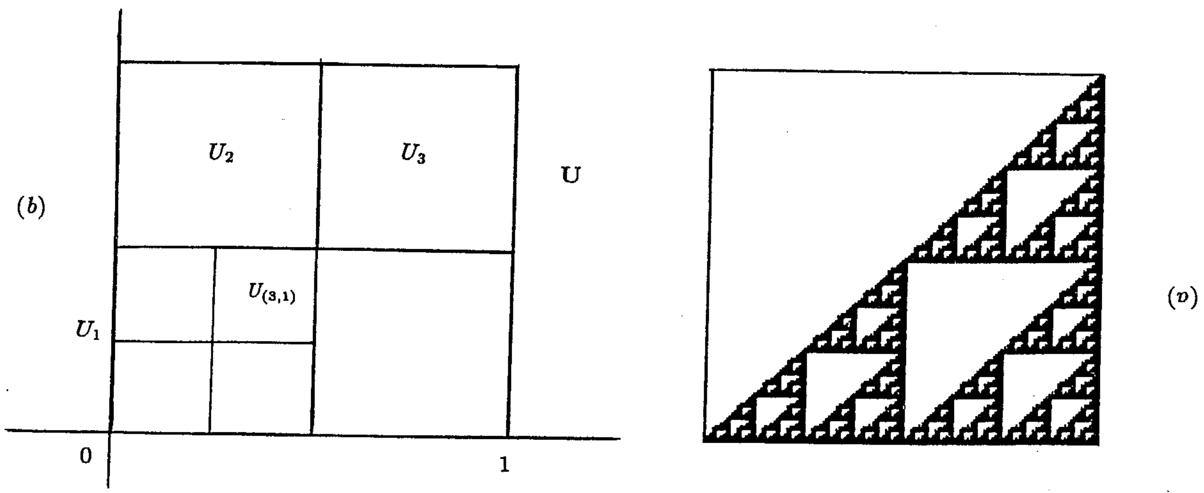
The contractive similarities are

$$S_1(x) = \frac{1}{3}x, \qquad S_2(x) = \frac{1}{3}e^{i\pi/3}x + \left(\frac{1}{3}, 0\right),$$

$$S_3(x) = \frac{1}{3}e^{-i\pi/3}x + \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \qquad S_4(x) = \frac{1}{3}x + \left(\frac{2}{3}, 0\right)$$

for $x \in \mathbf{R}^2$. Figure (a) is the invariant set E which supports the self-similar measure μ for any positive weights. The open set condition holds with the open set U taken to be the large triangle with base (0, 1), as shown in Figure (b). We can see that $\overline{U_{(2,1,4)}} \subseteq U$.

Example 2 (Sierpinski Triangle).



The contractive similarities are

$$S_1(x) = \frac{1}{2}x, \quad S_2(x) = \frac{1}{2}x + \left(0, \frac{1}{2}\right), \quad S_3(x) = \frac{1}{2}x + \left(\frac{1}{2}, \frac{1}{2}\right)$$

for $x \in \mathbf{R}^2$. The open set condition holds with the open set U taken as the unit square. In this case $\overline{U_{(1,3)}} \subseteq U$.

The additional condition $\mu(\partial U) = 0$ (equivalently, $\mu(U) = 1$) on the open set condition will be used in next section. To conclude this section we will give a few useful lemmas in connection to the multi-indices. For any $0 < t < 1$, and for any contractive similarities $\{S_j\}_{j=1}^m$ with contractive constants $\{\varrho_i\}_{i=1}^m$, we let

$$\Lambda(t) = \{J \in \Lambda \colon \varrho t < \varrho_J \leqslant t\},$$

where $\varrho = \min\{\varrho_j \colon j = 1, \ldots, m\}$. Note that for each infinite sequence $(j_1, j_2, \ldots)$ with $j_i = 1, \ldots, m$, $i = 1, 2, \ldots$, there is one and only one k such that $(j_1, \ldots, j_k) \in \Lambda(t)$.

Let μ be a self-similar measure satisfying (2.1). It is clear that μ_J is supported by $\overline{U_J}$ so that μ is supported by $\bigcup\{\overline{U_J} \colon |J| = k\}$ and $\sum_{|J|=k} a_J = 1$. The following lemma is used in [H] and [S2] without proof. For completeness we include a simple proof.

Lemma 2.8. *For any* $0 < t < 1$, μ *is supported by* $\bigcup\{\overline{U}_J\colon J \in \Lambda(t)\}$ *and* $\sum_{J\in\Lambda(t)} a_J = 1$.

Proof. Note that $\Lambda(t)$ is a finite set, let $K = \max\{|J|\colon J \in \Lambda(t)\}$. For any $J \in \Lambda(t)$ appeared in the sum, if $|J| < K$ we replace a_J by $\sum_{j=1}^m a_{(j,J)}$, the sum is not changed since $\sum_{j=1}^m a_{(j,J)} = a_J \sum_{j=1}^m a_j = a_J$. Continue this process and note that each $a_{J'}$, $|J'| = K$ must come from one of the J, $J \in \Lambda$. Eventually we have $\sum_{J\in\Lambda(t)} = \sum_{|J'|=K} a_{J'} = 1$. Q.E.D.

The following Lemma is a slight modification of a lemma of HUTCHINSON [H].

Lemma 2.9. *Suppose that* $\{S_j\}_{j=1}^m$ *satisfies the open set condition with an open set* U, *then for any* $c > 0$, *there exists an integer* K *such that for any* $0 < t < 1$ *and for any* $J \in \Lambda(t)$, *there are at most* K *of* $J' \in \Lambda(t)$ *with*

$$d(U_J, U_{J'}) \leqslant c\,t.$$

Proof. Without loss of generality we assume that U is contained in a ball of radius 1, and U contains a ball of radius r_0. For each $J \in \Lambda(t)$, U_J is contained in a ball of radius t and contains a ball of radius $r_0\,\varrho\,t$. Suppose there are q of $J' \in \Lambda(t)$ such that $d(U_J, U_{J'}) \leqslant c\,t$, they are hence all contained in a ball of radius $(3 + c)\,t$, and each of them contains a ball of radius $r_0\,\varrho\,t$. Note that all the U_J's are disjoint by the open set condition, summing up the volumes, we have

$$q\,(r_0\,\varrho\,t)^n \leqslant (3 + c)^n\,t^n.$$

The lemma follows by choosing $K = [((3 + c)/c_0\,\varrho)^n] + 1$. Q.E.D.

Lemma 2.10. *Suppose further* $\mu(\partial U) = 0$, *then*:
(i) *There exists* $0 < t_0 < \varrho$, *and* $J_0 \in \Lambda(t_0)$ *such that*

$$d(S_{J_0} U, \partial U) > 2\,t_0/\varrho.$$

(ii) *If* $J \in \Lambda(t)$ *can be written as* $J = (J_1, J_0, J_2)$ *for some multi-indices* J_1, J_2 *(they may be empty), then*

$$d(S_J U, \partial U) > 2\,t.$$

Remark. Part(i) asserts that if $\mu(\partial U) = 0$, we can find at least one $S_{J_0} U$ sufficiently far away from the boundary. The converse of this statement is also true in view of Corollary 2.4 (since in this case $S_{J_0}(\overline{U}) \subseteq U$). This lemma is crucial in the estimation in Lemma 3.6.

Proof. (i) Suppose $\mu(\partial U) = 0$, we claim that $d(U_J, \partial U) > 0$ for some $J \in \Lambda(t)$. Otherwise $d(U_J, \partial U) = 0$ for any $J \in \Lambda(t)$ and $0 < t < 1$. Since $E \subseteq \bigcup_{J \in \Lambda(t)} \overline{U}_J$, it follows that for any $x \in E$ and $t > 0$,

$$d(x, \partial U) \leqslant \sup_{J \in \Lambda(t)} \operatorname{diam}(\overline{U}_J) \leqslant t \cdot \operatorname{diam} U.$$

This implies that $x \in \partial U$, hence $E \subseteq \partial U$ and $\mu(\partial U) = 1$, that is a contraction.

By the claim, we now assume $d(U_{J'_0}, \partial U) > 0$ for some $J'_0 \in \Lambda$. Let

$$0 < t_0 = \min\left\{\frac{\varrho}{2} d(U_{J'_0}, \partial U),\ \varrho_{J'_0}\right\}.$$

and let $j \in \{1, \ldots, m\}$ be such that $\varrho_j = \min\{\varrho_1, \ldots, \varrho_m\}$. We can obtain J_0 by adding j repeatedly to J'_0 so that $J_0 = (j, \ldots, j, J'_0) \in \Lambda(t_0)$. Thus

$$\frac{2\,t_0}{\varrho} < d(U_{J'_0}, \partial U) \leqslant d(U_{J_0}, \partial U).$$

(ii) For $J \in \Lambda(t)$ with $J = (J_1, J_0, J_2)$, we have $\varrho_J = \varrho_{J_1} \varrho_{J_0} \varrho_{J_2} > \varrho t$. That $\varrho_{J_0} \leqslant t_0$ (since $J_0 \in \Lambda(t_0)$) implies that

$$\varrho_{J_1} \geqslant \varrho_{J_1} \varrho_{J_2} \geqslant \varrho\, t/\varrho_{J_0} \geqslant \varrho\, t/t_0.$$

Hence

$$\begin{aligned} d(S_J U, \partial U) &= d(S_{J_1} \circ S_{J_0} \circ S_{J_2}(U), \partial U) \geqslant \\ &\geqslant d(S_{J_1} \circ S_{J_0}(U), \partial U) \geqslant d(S_{J_1} \circ S_{J_0}(U), S_{J_1}(\partial U)) = \\ &= \varrho_{J_1} d(S_{J_0}(U), \partial U) \geqslant \varrho_{J_1} \cdot 2\, t_0/\varrho \geqslant 2\,t. \quad \text{Q.E.D.} \end{aligned}$$

3. Mean Quadratic Variations

For any positive σ-finite Borel measure ν on $\mathbf{R}^n$, we let, for $0 < t < 1$,

$$V_\alpha(t; \nu) = \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} \nu^2(B_t(x))\, dx,$$

and

$$V_\alpha(\nu) = \limsup_{t \to 0} V_\alpha(t; \nu).$$

We call $V_\alpha(\nu)$ the *α-upper mean quadratic variation* of ν; β is called the *m.q.v. index* of ν if

$$V_\alpha(\nu) = \begin{cases} \infty, & \text{if } \alpha > \beta; \\ 0, & \text{if } \alpha < \beta. \end{cases}$$

In the following we will relate the *m.q.v* index with a more familiar quantity. Recall that the *α-energy* of a probability measure ν is defined by

$$I_\alpha(\nu) = \iint \frac{d\nu(x)\, d\nu(y)}{|x-y|^\alpha}.$$

Lemma 3.1. *There exist $c_1, c_2 > 0$ such that for any probability measure ν on $\mathbf{R}^n$,*

$$\frac{c_1}{t^\alpha}\iint_{|\xi-\eta|\leqslant t} d\nu(\xi)\, d\nu(\eta) \leqslant V_\alpha(t;\nu) \leqslant \frac{c_2}{t^\alpha}\iint_{|\xi-\eta|\leqslant 2t} d\nu(\xi)\, d\nu(\eta).$$

Proof. Using Fubini's Theorem, we have

$$\begin{aligned} V_\alpha(t;\nu) &= \frac{1}{t^{n+\alpha}}\int\left[\iint \chi_{B_t(x)}(\xi)\,\chi_{B_t(x)}(\eta)\, d\nu(\xi)\, d\nu(\eta)\right] dx \\ &= \frac{1}{t^{n+\alpha}}\iint \omega_n(B_t(\xi)\cap B_t(\eta))\, d\nu(\xi)\, d\nu(\eta) \end{aligned}$$

where χ_B is the indicator function of set B and ω_n is the Lebesgue measure on R^n. Note that $B_t(\xi)\cap B_t(\eta)$ contains a ball of radius $t/2$ if $|\xi-\eta|\leqslant t$. The lemma follows by taking $c_1 = c/2^n$, and $c_2 = 2^n c$ where c is the volume of the ball of radius 1. Q.E.D.

Proposition 3.2. *Let ν be a probability measure on R^n.*
(i) *If $V_\beta(\nu) > 0$, then $I_\alpha(\nu) = \infty$ for all $\alpha > \beta$;*
(ii) *If $V_\beta(\nu) < \infty$, then $I_\alpha(\nu) < \infty$ for all $\alpha < \beta$.*

Proof. By Lemma 3.1, we have for any $0 < t < 1$,

$$V_\alpha(t;\nu) \leqslant 2^\alpha c_2 \iint \frac{1}{|\xi-\eta|^\alpha}\, d\nu(\xi)\, d\nu(\eta) = 2^\alpha c_2 I_\alpha(\nu)$$

and (i) follows. To prove (ii), we fix $0 < \lambda < 1$, then

$$I_\alpha \leqslant 1 + \sum_{m=0}^{\infty} \iint_{\lambda^{m+1} \leqslant |\xi - \eta| < \lambda^m} \frac{1}{|\xi - \eta|^\alpha} d\nu(\xi)\, d\nu(\eta) \leqslant$$

$$\leqslant 1 + \sum_{m=0}^{\infty} \frac{1}{\lambda^{(m+1)\alpha}} \iint_{|\xi - \eta| \leqslant \lambda^m} d\nu(\xi)\, d\nu(\eta) \leqslant$$

$$\leqslant 1 + (c_1 \lambda^\alpha)^{-1} \sum_{m=0}^{\infty} \lambda^{m(\beta - \alpha)} V_\beta(\nu; \lambda^m) \leqslant$$

$$\leqslant 1 + (c_1 \lambda^\alpha)^{-1} \left(\sum_{m=0}^{\infty} \lambda^{m(\beta - \alpha)} \right) \sup_{0 < t < 1} V_\beta(\nu; t) < \infty. \quad \text{Q.E.D.}$$

For any Borel subset E, we use $\dim(E)$ to denote the Hausdorff dimension of E. It is known that

$$\dim(E) = \sup\{\alpha\colon I_\alpha(\nu) < \infty,\ \mathrm{supp}(\nu) \subseteq E,\ \nu \text{ prob. meas.}\}.$$

As a direct consequence we have the following result.

Corollary 3.3. *Let E be a Borel subset in R^n, then*

$$\dim(E) = \sup\{\alpha\colon V_\alpha(\nu) < \infty,\ \mathrm{supp}(\nu) \subseteq E,\ \nu \textit{ prob. meas.}\}.$$

For the contractive similarities $\{S_j\colon j = 1, \ldots, m\}$ and weights $\{a_j\colon j = 1, \ldots, m\}$, we let β be the unique real number satisfying

$$\sum_{j=1}^{m} a_j^2 \varrho_j^{-\beta} = 1.$$

Proposition 3.4. *Suppose that $\{S_j\}_{j=1}^m$ satisfies the open set condition. Then $0 \leqslant \beta \leqslant n$ and $\beta = n$ if and only if $a_j = \varrho_j^n$. In this case the self-similar measure ν equals a multiple of the Lebesgue measure restricted on E.*

Proof. That $\beta \geqslant 0$ is trivial. Suppose that $\beta \geqslant n$, then $\varrho_j^\beta \leqslant \varrho_j^n$. The open set condition implies that $\bigcup_{j=1}^m U_j \subseteq U$ and U_j are disjoint, taking the volumes we have that $\sum_{j=1}^m \varrho_j^n \leqslant 1$. Hence

$$1 = \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} (a_j \varrho_j^{-\frac{\beta}{2}}) \varrho_j^{\frac{\beta}{2}} \leqslant \left(\sum_{j=1}^{m} a_j^2 \varrho_j^{-\beta} \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \varrho_j^\beta \right)^{\frac{1}{2}} \leqslant \left(\sum_{j=1}^{m} \varrho_j^n \right)^{\frac{1}{2}} \leqslant 1.$$

This implies that $\sum \varrho_j^\beta = \sum \varrho_j^n = 1$ and $\beta = n$. Using the equality case in the Schwartz Inequality, we conclude that $a_j \varrho_j^{-\frac{n}{2}} = \varrho_j^{\frac{n}{2}}$, i.e. $a_j = \varrho_j^n$. The second part follows from ([H], p. 737, Theorem 1 (iii)). Q.E.D.

Let $\Lambda(t)$ be defined as in Section 2, then the same argument as in Lemma 2.8 yields

$$\sum_{J\in\Lambda(t)} a_J^2 \varrho_J^{-\beta} = 1.$$

As a consequence for any $0 < t < 1$ we have

$$\varrho^\beta < \frac{1}{t^\beta} \sum_{J\in\Lambda(t)} a_J^2 \leqslant 1. \tag{3.1}$$

Theorem 3.5. *Suppose that $\{S_j\}_{j=1}^m$ satisfies the open set condition with respect to an open set U, and suppose that μ is a self-similar measure with $\mu(\partial U_i \cap \partial U_j) = 0$, for all $i \neq j$, then*

$$0 < \inf_{0<t<1} V_\beta(t;\mu) \leqslant \sup_{0<t<1} V_\beta(t;\mu) < \infty.$$

In particular the m.q.v. index of μ is β.

Proof. If $\beta = 0$, then $a_j = 1$ for some j, and $a_i = 0$ if $i \neq j$, so μ is a point mass measure and $V_\beta(t;\mu) = c \neq 0$. Without loss of generality we assume that the diameter of U is 1 and $0 < \beta \leqslant n$. Note that $\operatorname{supp}(\mu)$ is contained in $\bigcup\{\overline{U}_J: J\in\Lambda(t)\}$, and $\mu(\overline{U}_J) = a_J$ (Proposition 2.1), we have

$$V_\beta(t;\mu) = \frac{1}{t^{n+\beta}} \int\int \omega_n(B_t(\xi)\cap B_t(\eta))\, d\mu(\xi)\, d\mu(\eta) \geqslant$$

$$\geqslant \frac{1}{t^{n+\beta}} \sum_{J\in\Lambda(t)} \int\int_{\xi,\,\eta\in\overline{U}_J} \omega_n(B_t(\xi)\cap B_t(\eta))\, d\mu(\xi)\, d\mu(\eta).$$

Note that $\operatorname{diam}(\overline{U_J}) = \varrho_J \leqslant t$, $B_t(\xi)\cap B_t(\eta)$ contains a ball of radius $t/2$ whenever $\xi, \eta \in \overline{U}_J$. It follows that

$$V_\beta(t;\mu) \geqslant \frac{c}{t^\beta} \sum_{J\in\Lambda(t)} \int\int_{\xi,\,\eta\in\overline{U}_J} d\mu(\xi)\, d\mu(\eta) \geqslant \frac{c}{t^\beta} \sum_{J\in\Lambda(t)} a_J^2 \geqslant c\,\varrho^\beta,$$

where c depends only on n, and the first inequality follows.

To prove the second inequality we let, for each $J\in\Lambda(t)$,

$$\Lambda(t;J) = \{J'\in\Lambda(t): d(U_J, U_{J'}) \leqslant 2t\}.$$

By Lemma 2.9, $\Lambda(t;J)$ can have at most K members where K is independent of t and J. Hence by Lemma 3.1,

$$V_\beta(t;\mu) \leqslant \frac{c}{t^\beta} \iint \chi_{\{|\xi-\eta|\leqslant 2t\}}\, d\mu(\xi)\, d\mu(\eta) =$$

$$= \frac{c}{t^\beta} \sum_{J\in\Lambda(t)} \int_{\xi\in\bar{U}_J} \int_{\eta\in\mathbf{R}^n} \chi_{\{|\xi-\eta|\leqslant 2t\}}\, d\mu(\xi)\, d\mu(\eta) \leqslant$$

$$\leqslant \frac{c}{t^\beta} \sum_{\substack{J, J'\in\Lambda(t)\\ d(U_J, U_{J'})\leqslant 2t}} \int_{\xi\in\bar{U}_J} \int_{\eta\in\bar{U}_{J'}} d\mu(\xi)\, d\mu(\eta) \leqslant \tag{3.2}$$

$$\leqslant \frac{c}{t^\beta} \sum_{\substack{J, J'\in\Lambda(t)\\ d(U_J, U_{J'})\leqslant 2t}} a_J a_{J'} \leqslant \frac{c}{2t^\beta} \sum_{\substack{J, J'\in\Lambda(t)\\ d(U_J, U_{J'})\leqslant 2t}} (a_J^2 + a_{J'}^2) \leqslant$$

$$\leqslant \frac{c}{2t^\beta}\Big(\sum_{J\in\Lambda(t)} \sum_{J'\in\Lambda(t,J)} a_J^2 + \sum_{J'\in\Lambda(t)} \sum_{J\in\Lambda(t;J')} a_{J'}^2\Big) \leqslant$$

$$\leqslant \frac{cK}{t^\beta} \sum_{J\in\Lambda(t)} a_J^2 \leqslant c_2 K. \qquad \text{(by (3.1))}$$

This completes the proof of the theorem. Q.E.D.

We now turn to a more accurate estimation of $V_\beta(\mu; t)$ as $t\to 0$ under a slight stronger condition, we first observe that

$$V_\beta(t;\mu) = \frac{1}{t^{n+\beta}} \int \Big|\sum a_j \mu_j(B_t(x))\Big|^2 dx =$$

$$= \frac{1}{t^{n+\beta}} \sum a_j^2 \int \mu_j^2(B_t(x))\, dx + E(t) =$$

$$= \frac{1}{t^{n+\beta}} \sum a_j^2 \int \mu^2(B_{\varrho_j^{-1}t}(y))\, \varrho_j^n\, dy + E(t) = \qquad (x = S_j(y))$$

$$= \sum a_j^2 \varrho_j^{-\beta} V_\beta(\varrho_j^{-1} t;\mu) + E(t), \tag{3.3}$$

where $E(t) = \sum_{i\neq j} a_i a_j E_{ij}(t)$ with

$$E_{ij}(t) = \frac{1}{t^{n+\beta}} \int \mu_i(B_t(x))\, \mu_j(B_t(x))\, dx.$$

The main estimation is the following result.

Lemma 3.6. *Suppose that $\{S_j\}_{j=1}^m$ satisfies the open set condition with an open set U and $\mu(\partial U) = 0$, then $E(t) = O(t^\varepsilon)$ as $t \to 0$ for some $\varepsilon > 0$.*

Remark. If the strong open set condition holds, then $\min\{d(\overline{U_i}, \overline{U_j}) : i \neq j\} > 0$. Since μ_i, μ_j has support contained in $\overline{U_i}$ and $\overline{U_j}$ respectively, $\mu_i(B_t(x))\mu_j(B_t(x)) = 0$ for t sufficiently small, and the lemma is trivial.

Proof. Using the same estimation as in (3.2) we have

$$E_{ij}(t) \leqslant \frac{c}{t^\beta} \sum_{J, J'} \int_{\xi \in S_i(\overline{U}_J)} \int_{\eta \in S_j(\overline{U}_{J'})} d\mu_i(\xi)\, d\mu_j(\eta) = \frac{c}{t^\beta} \sum_{J, J'} a_J a_{J'},$$

where the sum is taken over all pairs (i, J), $(j, J') \in \Lambda(t)$ such that

$$d(S_i(U_J), S_j(U_{J'})) \leqslant 2t.$$

Noting that $S_i(U_J) \subseteq S_i(U)$, and $S_j(U_{J'})$ is disjoint from the interior of $S_i(U)$, we have

$$d(S_i(U_J), \partial(S_i U)) \leqslant d(S_i(U_J), S_j(U_{J'})) \leqslant 2t.$$

Hence

$$d(U_J, \partial U) \leqslant 2t/\varrho_i;$$

the same result holds for J'. We have then by the above and Lemma 2.9 that

$$E_{ij}(t) \leqslant \frac{cK}{2t^\beta} \Bigg(\sum_{\substack{(i, J) \in \Lambda(t) \\ d(U_J, \partial U) \leqslant 2t/\varrho_i}} + \sum_{\substack{(j, J) \in \Lambda(t) \\ d(U_J, \partial U) \leqslant 2t/\varrho_j}} \Bigg) a_J^2 \leqslant$$

$$\leqslant \frac{cK}{2t^\beta} \Bigg(\sum_{\substack{J \in \Lambda(t/\varrho_i) \\ d(U_J, \partial U) \leqslant 2t/\varrho_i}} + \sum_{\substack{J \in \Lambda(t/\varrho_j) \\ d(U_J, \partial U) \leqslant 2t/\varrho_j}} \Bigg) a_J^2$$

(see the argument in theorem 3.5). Set

$$W(t) = t^{-\beta} \sum_{\substack{J \in \Lambda(t) \\ d(U_J, \partial U) \leqslant 2t}} a_J^2.$$

We show that $W(t) = O(t^\varepsilon)$ as $t \to 0$, and the proof of the lemma will be complete. Let t_0 and J_0 be defined as in Lemma 2.10, we use $J_0 \not\prec J$ to mean that J_0 does not equal to any segment of J. It follows from Lemma 2.10(ii) that

$$W(t) \leqslant \frac{1}{t^{\beta}} \sum_{\substack{J \in \Lambda(t) \\ J_0 \not\prec J}} a_J^2. \tag{3.4}$$

Let $t < t_0$, and let n be a positive integer such that

$$t_0^{n+1} < t \leqslant t_0^n .$$

For $J \in \Lambda(t)$, we can write $J = (J_1, J_1')$ where $J_1 \in \Lambda(t_0)$, i.e.,

$$\varrho t_0 < \varrho_{J_1} \leqslant t_0 .$$

Inductively we can write $J = (J_1, \ldots, J_k, J')$, $J_i \in \Lambda(t_0)$, $i = 1, \ldots, k$ with $k = \left[\frac{n}{2}\right]$. This is possible since

$$\varrho_{J_1 \ldots J_k} = \varrho_{J_1} \ldots \varrho_{J_k} \geqslant (\varrho t_0)^k > t_0^{2k} \geqslant t .$$

It is also obvious that for any $J_1, J_2, \ldots, J_k \in \Lambda(t_0)$, there exist J' such that $(J_1, \ldots, J_k, J') \in \Lambda(t)$. In fact, for each such $(J_1, \ldots, J_k)$,

$$\{J' : (J_1, \ldots, J_k, J') \in \Lambda(t)\} = \Lambda(t/\varrho_{J_1} \ldots \varrho_{J_k}).$$

We hence have by (3.4),

$$\begin{aligned} W(t) &\leqslant \frac{1}{t^{\beta}} \sum_{J_0 \not\prec J \in \Lambda(t)} a_J^2 \leqslant \sum_{J_0 \not\prec J \in \Lambda(t)} a_J^2 \varrho_J^{-\beta} \leqslant \\ &\leqslant \sum_{\substack{J_0 \neq J_j \\ J_1, \ldots, J_k \in \Lambda(t_0)}} \left[a_{J_1}^2 \varrho_{J_1}^{-\beta} \cdot a_{J_2}^2 \varrho_{J_2}^{-\beta} \ldots a_{J_k}^2 \varrho_{J_k}^{-\beta} \cdot \sum_{J' \in \Lambda(t/\varrho_{J_1} \ldots \varrho_{J_k})} a_{J'}^2 \varrho_{J'}^{-\beta} \right] = \\ &= \left(\sum_{J_0 \neq J_1 \in \Lambda(t_0)} a_{J_1}^2 \varrho_{J_1}^{-\beta} \right) \left(\sum_{J_0 \neq J_2 \in \Lambda(t_0)} a_{J_2}^2 \varrho_{J_2}^{-\beta} \right) \ldots \left(\sum_{J_0 \neq J_k \in \Lambda(t_0)} a_{J_k}^2 \varrho_{J_k}^{-\beta} \right) \cdot 1 = \\ &= \left(\sum_{J_0 \neq J \in \Lambda(t_0)} a_J^2 \varrho_J^{-\beta} \right)^k \leqslant c^{\ln t/2 \ln t_0} = t^{\varepsilon}, \end{aligned}$$

where $c = (\sum_{J_0 \neq J \in \Lambda(t_0)} a_J^2 \varrho_J^{-\beta}) < 1$, so that $\varepsilon > 0$. Q.E.D.

Now we state the main theorem.

Theorem 3.7. *Let β be such that $\sum_{j=1}^m a_j^2 \varrho_j^{-\beta} = 1$. Suppose $\mu(\partial U) = 0$ for some U satisfying the open set condition.* Then

$$\lim_{t \to 0} (V_\beta(t; \mu) - P(t)) = 0$$

for some $P > 0$ such that the following holds.

(i) If $\{-\ln \varrho_j : j = 1, \ldots, m\}$ *is non-arithmetic, then* $P(x) = c$ *for some constant* c.

(ii) *Otherwise, let* $((\ln \lambda)\mathbf{Z})$, $\lambda > 1$ *be the lattice generated by* $\{-\ln \varrho_j : j = 1, \ldots, m\}$, *then* $P(\lambda t) = P(t)$.

The proof relies on the following well known theorem on the renewal equation **[Fe]**. We will call a function h *directly Riemann integrable* if $\sum M_k$, $\sum m_k$ converge absolutely for any $s > 0$ and

$$\sum_{k=-\infty}^{\infty} (M_k - m_k) s \to 0 \quad \text{as} \quad s \to 0, \tag{3.5}$$

where $M_k = \sup\{h(x) : x \in [ks, (k+1)s]\}$, $m_k = \inf\{h(x) : x \in [ks, (k+1)s]\}$. Note that this definition is stronger than the usual way of defining Riemann integrable function on the infinite domain $[0, \infty)$.

By a direct check, we can see that the property h is directly Riemann integrable is equivalent to h is locally Riemann integrable and

$$\sum \| h \chi_{[n, n+1]} \|_\infty < \infty .$$

This class will play a more important role in next section.

Theorem 3.8. *Let* $\sigma \neq \delta_0$ *be a probability measure on* $[0, \infty)$ *and let* S *be a bounded measurable function on* $[0, \infty)$. *Suppose that* f *satisfies the renewal equation*

$$f(x) = \int_0^x f(x-y)\, d\sigma(y)\, d\sigma(y) + S(x), \qquad x \geq 0,$$

then $f = \sum_{n=0}^{\infty} S * \sigma^n$. *If in addition,* S *is directly Riemann integrable, then the following holds.*

(i) *If* σ *is non-arithmetic, then* $f(x) = c + o(1)$ *as* $x \to \infty$, *where* $c = [\int_0^\infty y\, d\sigma(y)]^{-1} \int_0^\infty S(y)\, d\sigma(y)$.

(ii) *If* σ *is arithmetic and* $\operatorname{supp}(\sigma)$ *generates a lattice* $\lambda \mathbf{Z}$, $\lambda > 0$, *then* $\varphi(x) = p(x) + o(1)$ *as* $x \to \infty$ *where* $p(x) = \lambda [\int_0^\infty y\, d\sigma(y)]^{-1} \cdot \sum_k S(x + k\lambda)$ *is a periodic function with period* λ.

Proof of Theorem 3.7. Let σ be the atomic measure defined by $\sigma\{\varrho_j\} = a_j^2 \varrho_j^{-\beta}$, $j = 1, \ldots, m$, then by (3.3),

$$V_\beta(t; \mu) = \int_0^1 V_\beta(t/s; \mu)\, d\sigma(s) + E(t), \qquad 1 > t > 0.$$

Now let $\xi = -\ln t$, $\eta = -\ln s$ and $f(\xi) = V_\beta(\mu; t)$, we can reduce the above equation to

$$f(\xi) = \int_0^\infty f(\xi - \eta)\, d\tilde{\sigma}(\eta) + E(e^{-\xi}), \qquad \xi \geqslant 0,$$

where $\tilde{\sigma}(\eta) = \sigma(e^{-\eta})$. Let $S(\xi) = \int_\xi^\infty f(\xi - \eta)\, d\tilde{\sigma}(\eta) + E(e^{-\xi})$, then

$$f(\xi) = \int_0^\xi f(\xi - \eta)\, d\tilde{\sigma}(\eta) + S(\xi), \qquad \xi \geqslant 0.$$

The fact that $\tilde{\sigma}$ has compact support implies that $\int_\xi^\infty f(\xi - \eta)\, d\tilde{\sigma}(\eta) = 0$ for ξ large, so that $S(\xi) = O(e^{-\varepsilon\xi})$ (Lemma 3.6). Hence $S(\xi)$ and σ satisfy the conditions of the renewal equation, and Theorem 3.8 applies. That $c > 0$ and $P > 0$ follows directly from Theorem 3.5. Q.E.D.

4. Wiener Plancherel Formula and Tauberian Theorems

In order to transfer the results of the mean quadratic variations to Fourier asymptotics of self-similar measures μ, we need some special forms of Tauberian theorems. In this section we will proceed with the following presentation, which covers a larger class of functions other than $\hat{\mu}$. It generalizes, in some sense, the Wiener-Plancherel formula in WIENER's generalized harmonic analysis **[Wi]** and also the previous works in **[LL]** and **[CL]**. Let

$$J_k(r) = \frac{(r/2)^k}{\Gamma((2k+1)/2)\,\Gamma(1/2)} \int_{-1}^1 e^{irs}(1 - s^2)^{(2k-1)/2}\, ds, \qquad r \geqslant 0$$

$\left(\text{with } k \geqslant -\frac{1}{2}\right)$ be the Bessel function of order k. It is clear that $\lim_{r\to 0} J_k(r)/r^k = c > 0$, also for $k > 0$, $J_k(r) = O(r^{-1/2})$ as $r \to \infty$ (**[SW]**, p. 158). Set

$$E_t(x) = \int_{B_t} e^{-2\pi i x \xi}\, d\xi \qquad \forall x \in R^n,$$

where $B_t = B_t(0)$ is the ball centered at 0 with radius t. The following can be easily checked (**[SW]**, **[Wa]**).

Proposition 4.1. *Let E_t be defined as above.*

(i) $(\chi_{B_t})\hat{} = E_t$, *where* χ_{B_t} *is the indicator function of* B_t.

(ii) $E_t(x) = (t/|x|)^{\frac{n}{2}} J_{\frac{n}{2}}(2\pi t|x|)$.

(iii) E_t *is bounded*, $\lim_{|x|\to 0} E_t(x) > 0$, *and* $E_t(x) = O\left(|x|^{-\frac{n+1}{2}}\right)$ *as* $|x| \to \infty$.

For $1 \leqslant p < \infty$ and $0 \leqslant \alpha \leqslant n$, we use $B^p_\alpha(\mathbf{R}^n)$ to denote the space of functions of $f \in L^p_{loc}(\mathbf{R}^n)$ such that

$$\sup_{T\geqslant 1} \frac{1}{T^{n-\alpha}} \int_{B_T} |f|^p < \infty.$$

Proposition 4.2. *For* $1 \leqslant p < \infty$, $0 \leqslant \beta \leqslant \alpha \leqslant n$, *and* $\delta > 0$, *we have*

$$B^p_\alpha \subseteq B^p_\beta \subseteq B^p_0 \subseteq L^p\left(\frac{dx}{1+|x|^{n+\delta}}\right).$$

Proof. The first two inclusions are obvious. The last inclusion follows from the fact that $f \in B^p_\alpha$ and

$$\int_{\mathbf{R}^n} \frac{|f(x)|^p}{1+|x|^{n+\delta}}\, dx \leqslant \int_{|x|\leqslant 1} |f|^p + \sum_{k=1}^{\infty} \frac{1}{1+2^{k(n+\delta)}} \int_{2^k<|x|\leqslant 2^{k+1}} |f|^p \leqslant$$

$$\leqslant \left(1 + \sum_{k=1}^{\infty} \frac{2^{(n-\alpha)(k+1)}}{1+2^{k(n+\delta)}}\right) \cdot \sup_{T\geqslant 1} \frac{1}{T^{n-\alpha}} \int_{|x|\leqslant T} |f|^p < \infty. \quad \text{Q.E.D.}$$

For any $f \in B^2_\alpha(\mathbf{R}^n)$, we let

$$W_t(f)(y) = \int_{R^n} f(x)\, E_t(x)\, e^{2\pi i x y}\, dx = (f \cdot E_t)\check{}\,(y)$$

be the Wiener's transformation of f, where $\check{g}$ denotes the inverse Fourier transformation of g. By Proposition 4.1(iii), $f \cdot E_t \in L^2(\mathbf{R}^n)$ and hence $W_t(f)$ is well-defined on $B^2_\alpha(\mathbf{R}^n)$. The main motivation of introducing the transformation W_t is the following simple identity.

Proposition 4.3. *Let* μ *be a bounded measure on* $\mathbf{R}^n$, *and let* $f = \hat{\mu}$, *then for* $t > 0$ *and for almost all* y *(with respect to the Lebesgue measure)*

$$W_t(f)(y) = \mu(B_t(y)).$$

Proof. If follows from the definition that $W_t(f)\hat{} = f \cdot E_t = \hat{\mu} \cdot E_t$ and $\mu(B_t(\cdot))\hat{} = (\mu * \chi_{B_t})\hat{} = \hat{\mu} \cdot E_t$. Q.E.D.

The following theorem (and also Theorem 4.10) allows us to transform results on the mean quadratic variations to the quadratic means of the Fourier transformation with respect to appropriate powers.

Theorem 4.4. *For* $f \in B^2_\alpha(\mathbf{R}^n)$,

$$\sup_{T \geqslant 1} \frac{1}{T^{n-\alpha}} \int_{|x| \leqslant T} |f(x)|^2\, dx \approx \sup_{0 < t \leqslant 1} \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |(W_t f)(y)|^2\, dy. \quad (4.1)$$

Moreover the pair $\sup_{T \geqslant 1}$, $\sup_{0 < t \leqslant 1}$ *can be replaced by the pair* $\limsup_{T \to \infty}$, $\limsup_{t \to 0}$ *respectively.*

Proof. Let $F(r) = r^\alpha \int_{S_{n-1}} |f(ru)|^2\, du$, then

$$\frac{1}{T^{n-\alpha}} \int_{|x| \leqslant T} |f(x)|^2\, dx = \frac{1}{T^{n-\alpha}} \int_0^T \left[\int_{S_{n-1}} |f(ru)|^2\, du \right] r^{n-1}\, dr =$$

$$= \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr. \quad (4.2)$$

Also letting $t = \frac{1}{T}$, $w(r) = r^{-n} J^2_{\frac{n}{2}}(2\pi r)$, we have

$$\frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |W_t f(y)|^2\, dy = \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |f(x)\, E_t(x)|^2\, dx =$$

$$= \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |f(x)|^2 \left((t/|x|)^{\frac{n}{2}} J_{\frac{n}{2}}(2\pi t |x|)\right)^2 dx = \int_0^\infty F(Tr)\, r^{n-\alpha-1}\, w(r)\, dr.$$
$$(4.3)$$

Hence to prove (4.1), we need only prove

$$\sup_{T \geqslant 1} \int_0^\infty F(Tr)\, r^{n-\alpha-1}\, w(r)\, dr \approx \sup_{T \geqslant 1} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr. \quad (4.4)$$

Note that $\lim_{r \to 0} J_k(r)/r^k = c > 0$. There exists a $0 < \lambda < 1$ such that $w(r) = r^{-n} J^2_{\frac{n}{2}}(2\pi r) > c_1 > 0$ if $0 < r < \lambda$. Thus

$$\int_0^\infty F(Tr)\, r^{n-\alpha-1}\, w(r)\, dr \geqslant c_1 \int_0^\lambda F(Tr)\, r^{n-\alpha-1}\, dr =$$

$$= c_2 \int_0^1 F(T\lambda r)\, r^{n-\alpha-1}\, dr \quad (4.5)$$

for $T > 1$, so that

$$\sup_{T \geqslant 1} \int_0^\infty F(Tr)\, r^{n-\alpha-1} w(r)\, dr \geqslant c_2 \sup_{T \geqslant 1} \int_0^1 F(T\lambda r)\, r^{n-\alpha-1}\, dr \geqslant$$

$$\geqslant c_2 \sup_{T \geqslant 1} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr. \qquad (4.6)$$

To prove the reverse inequality we let $\tilde{\omega}(r) = \sup_{s \geqslant r} w(s)$ be the smallest descreasing majorant of w. Since w is bounded and $w(r) = O(r^{-(n+1)})$ as $r \to \infty$, so is $\tilde{\omega}$. Now

$$\int_0^\infty F(Tr)\, r^{n-\alpha-1} w(r)\, dr \leqslant$$

$$\leqslant \tilde{\omega}(0) \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr + \sum_{k=0}^\infty \tilde{\omega}(2^k) \int_{2^k}^{2^{k+1}} F(Tr)\, r^{n-\alpha-1}\, dr \leqslant$$

$$\leqslant \tilde{\omega}(0) \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr +$$

$$+ \sum_{k=0}^\infty 2^{(k+1)(n-\alpha)} \tilde{\omega}(2^k) \int_0^1 F(T 2^{k+1} s)\, s^{n-\alpha-1}\, ds, \qquad (4.7)$$

by a change of variables $r = 2^{k+1} s$ on each term, so

$$\sup_{T \geqslant 1} \int_0^\infty F(Tr)\, r^{n-\alpha-1} w(r)\, dr \leqslant c_3 \sup_{T \geqslant 1} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr,$$

where $c_3 = \tilde{\omega}(0) + \sum_{k=0}^\infty 2^{(k+1)(n-\alpha)} \tilde{\omega}(2^k) \leqslant \tilde{\omega}(0) + 2\int_1^\infty \tilde{\omega}(r)\, r^{n-\alpha-1}\, dr < \infty$. This completes the proof of (4.4), and hence (4.1).

To show the lim sup case we see from (4.5) that for any $\varrho > 0$

$$\sup_{T \geqslant \varrho} \int_0^\infty F(Tr)\, r^{n-\alpha-1} w(r)\, dr \geqslant c_2 \sup_{T \geqslant \varrho} \int_0^1 F(T\lambda r)\, r^{n-\alpha-1}\, dr =$$

$$= c_2 \sup_{T \geqslant \varrho\lambda} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr \geqslant c_2 \sup_{T \geqslant \varrho} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr.$$

Hence

$$\limsup_{T \to \infty} \int_0^\infty F(Tr)\, r^{n-\alpha-1} w(r)\, dr \geqslant c_2 \limsup_{T \to \infty} \int_0^1 F(Tr)\, r^{n-\alpha-1}\, dr.$$

The reverse inequality follows from (4.7) in the same way. Q.E.D.

We remark that an analogous case of (4.4) has been proved in [CL], and the lim sup case in [LL]. The proofs there are intended to obtain the sharp isomorphic constants and hence more complicated; they can be used to prove the present case also.

Corollary 4.5. *Let μ be a bounded Borel measure on $\mathbf{R}^n$. Then*

$$\sup_{T \geqslant 1} \frac{1}{T^{n-\alpha}} \int_{|x| \leqslant T} |\hat{\mu}(x)|^2 \, dx \approx \sup_{0 < t \leqslant 1} \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |\mu(B_t(y))|^2 \, dy,$$

and

$$\limsup_{T \to \infty} \frac{1}{T^{n-\alpha}} \int_{|x| \leqslant T} |\hat{\mu}(x)|^2 \, dx \approx \limsup_{t \to 0} \frac{1}{t^{n+\alpha}} \int_{\mathbf{R}^n} |\mu(B_t(y))|^2 \, dy.$$

Proof. These are direct consequence of Proposition 4.3 and Theorem 4.4. Q.E.D.

The main purpose in the rest of this section is to prove a Tauberian Theorem to cover the periodic case which appears in Theorem 3.7(ii) (of course it will cover (i) automatically). Set

$$W(\mathbf{R}) = \{g \colon g \text{ continuous on } \mathbf{R}, \ \|g\| = \sum_{k=-\infty}^{\infty} \|g \chi_{[k,k+1)}\|_\infty < \infty\}.$$

Then $W(\mathbf{R})$ is a Banach space, whose dual is given by

$$E(\mathbf{R}) = \{\mu \colon \mu \text{ regular}, \ \|\mu\| = \sup_k |\mu|([k, k+1)) < \infty\}.$$

Note that any bounded Borel measurable f can be considered as an element in $E(\mathbf{R})$ with $\|f\|_{E(\mathbf{R})} = \sup_k \int_k^{k+1} |f(x)| \, dx$, and $\|f\|_{E(\mathbf{R})} \leqslant \|f\|_{L^\infty(\mathbf{R})}$.

Lemma 4.6. *Let A be a closed translation invariant subspace of $L^\infty(\mathbf{R})$ and let $g \in W(\mathbf{R})$ be such that $\hat{g}(\xi) \neq 0$ for all $\xi \in \mathbf{R}$. Suppose $\mu \in E(\mathbf{R})$ and $g * \mu \in A$. Then $h * \mu \in A$ for all $h \in W(\mathbf{R})$.*

Proof. Let $F = \{h \in W(\mathbf{R}) \colon h * \mu \in A\}$. It is clear that F is a translation invariant subspace of $W(\mathbf{R})$. To show that F is closed, we let $\{g_n\} \subseteq F$ and $g_i \to g$, then

$$|g_i * \mu(x) - g * \mu(x)| = \left| \sum_{k=-\infty}^{\infty} \int_k^{k+1} (g_i(y) - g(y))\, d\mu(x-y) \right| \leqslant$$
$$\leqslant \sum_{k=-\infty}^{\infty} \|(g_i - g)\chi_{[k,k+1)}\|_\infty \cdot |\mu|(x - [k, k+1)) \leqslant$$
$$\leqslant \|g_i - g\|_{W(\mathbf{R})} \cdot 2\, \|\mu\|_{E(\mathbf{R})}.$$

This implies that $g_i * \mu \to g * \mu$ in $L^\infty(\mathbf{R})$, since A is closed in $L^\infty(\mathbf{R})$, $g_i * \mu \in A$ implies that $g * \mu \in A$, so that $g \in F$. By Wiener's Second Tauberian Theorem ([T], Theorem 7.6), $F = W(\mathbf{R})$, i.e. $h \in \mu \in A$ for all $h \in W(\mathbf{R})$. Q.E.D.

If $A = \{f \in E(\mathbf{R}): \lim_{x\to\infty} f(x) = c$ for some $c \in \mathbf{C}\}$, then the above lemma is just the standard Tauberian Theorem in the limit form.

Theorem 4.7. *Let $g \in W(\mathbf{R})$ with $\hat{g}(\xi) \neq 0$ for all $\xi \in \mathbf{R}$. Let $\mu \in E(\mathbf{R})$ be such that*

$$\lim_{x\to\infty} (g * \mu(x) - p(x)) = 0$$

for some bounded periodic function of period a. Then for any $h \in W(\mathbf{R})$ there exists a bounded function q of period a such that

$$\lim_{x\to\infty} (h * \mu(x) - q(x)) = 0.$$

Moreover if we write $p(x) = \sum_{k=-\infty}^{\infty} a_k e^{i(2\pi k/a)x}$, $q(x) = \sum_{k=-\infty}^{\infty} b_k e^{i(2\pi k/a)x}$, then

$$b_k = a_k \frac{\hat{h}(2\pi k/a)}{\hat{g}(2\pi k/a)}, \qquad k \in \mathbf{N}.$$

Proof. Without loss of the generality we assume that $a = 2\pi$. Set

$$A = \{f \in L^\infty(\mathbf{R}): \lim_{x\to\infty} (f(x) - q(x)) = 0 \text{ for some } q \text{ with period } 2\pi\}.$$

then A is translation invariant. It is also closed in $L^\infty(\mathbf{R})$: for let $\{f_i\} \subseteq A$ with $f_i \to f$ in $L^\infty(\mathbf{R})$, then

$$\lim_{x\to\infty} (f_i(x) - q_i(x)) = 0 \qquad \text{for all } i,$$

for some bounded periodic functions $\{q_i\}$. For any $x \in \mathbf{R}$ and for any $\varepsilon > 0$,

$$|q_i(x) - q_j(x)| = |q_i(x + 2\pi k) - q_j(x + 2\pi k)| \leqslant$$
$$\leqslant |f_i(x + 2\pi k) - f_j(x + 2\pi k)| + 2\varepsilon. \qquad (4.8)$$

Taking k sufficiently large (k depends on i, j), this implies that $\{q_i\}$ is a Cauchy sequence, converges to a q in $L^\infty(\mathbf{R})$ of period 2π. We have $\lim_{x\to\infty}(f(x) - q(x)) = 0$, so that $f \in A$. Let $F = \{h \in W(\mathbf{R}): h * \mu \in A\}$, then $F = W(\mathbf{R})$ by Lemma 4.6 and the first conclusion follows.

To prove the second statement we let $h \in F = W(\mathbf{R})$ with the corresponding bounded 2π-periodic function q. By Wiener's Second Tauberian Theorem, there exists a sequence $\{\psi_j\} \subseteq L^1(\mathbf{R})$ such that $\psi_j * g \to h$ in $W(\mathbf{R})$, hence $\psi_j * g * \mu \to h * \mu$ in $L^\infty(\mathbf{R})$. Since $\lim_{j\to\infty}(\psi_j * g * \mu(x) - \psi_j(x) * p) = 0$, the same argument as in (4.8) implies that $\lim_{j\to\infty}(\psi_j * p) = q$ in $L^\infty(\mathbf{R})$. It follows that

$$b_k = \int_0^{2\pi} q(x)\, e^{-ikx}\, dx = \lim_{j\to\infty} \int_0^{2\pi} \psi_j * p(x)\, e^{-ikx}\, dx =$$

$$= \lim_{j\to\infty} a_k \cdot \hat{\psi}_j(k) = a_k \lim_{j\to\infty} \frac{(\psi_j * q)\hat{}\,(k)}{\hat{g}(k)} = a_k \cdot \frac{\hat{h}(k)}{\hat{g}(k)}. \qquad \text{Q.E.D.}$$

Remarks. 1. Note that if $p(x) \equiv c$, then q is also a constant function and $q \equiv b_0 = a_0/\hat{g}(0) = c/\hat{g}(0)$.

2. The continuity of g in the theorem can be replaced by the local integrability. In this case we can consider $\tilde{g} = e^{-x^2} * g$, note that $(\tilde{g})\hat{}\,(\xi) \neq 0$ for all $\xi \in \mathbf{R}$ and $\tilde{g} * A \subseteq A$, so the theorem applies.

3. For some applications it is more convenient to extend the conclusion to include discontinuous h also. We consider such extension in the following theorem.

Set

$$\tilde{W}(\mathbf{R}) = \{g\colon g \text{ is locally Riemann integrable, } \sum_{k=-\infty}^{\infty} \|g\chi_{[k,k+1]}\|_\infty < \infty\}.$$

Then $\tilde{W}$ is the class of directly Riemann integrable functions as defined in (3.5) for the renewal equation.

Theorem 4.8. *Let* $g \in \tilde{W}(\mathbf{R})$, $g \geqslant 0$ *and* $\hat{g}(\xi) \neq 0$ *for all* $\xi \in \mathbf{R}$. *Let* $\mu \geqslant 0$ *be a regular measure such that* $\limsup_{j\to-\infty} \mu[j, j+1) < \infty$ *and*

$$\lim_{x\to\infty} (g * \mu(x) - p(x)) = 0 \qquad (4.9)$$

for some bounded periodic function p with period a. Then for any $h \in \tilde{W}(\mathbf{R})$, *there exists a bounded periodic q of period a such that*

$$\lim_{x \to \infty} (h * \mu(x) - q(x)) = 0,$$

and the Fourier coefficients $\{a_k\}$ *and* $\{b_k\}$ *of p, q are related by*

$$b_k = a_k \frac{\hat{h}(2\pi k/a)}{\hat{g}(2\pi k/a)}, \qquad k \in \mathbf{N}.$$

Proof. By (4.9), we can show that $\limsup_{j \to \infty} \mu[j, j+1) < \infty$. Together with the assumption $\limsup_{j \to -\infty} \mu[j, j+1) < \infty$, and the regularity of μ, we have $\sup_j \mu[j, j+1) < \infty$ so that $\mu \in E(\mathbf{R})$.

Again we can assume that p has period 2π. It follows directly from the equivalent definition (3.5) of functions in $\tilde{W}(\mathbf{R})$ that there exist two sequences $\{g_j\}$, $\{f_j\}$ in $W(\mathbf{R})$ such that

$$g_j \searrow h, f_j \nearrow h, \text{ and } \lim_{j \to \infty} \int (g_j - f_j) = 0. \tag{4.10}$$

Let $\{p_j\}$ and $\{q_j\}$ denote the corresponding periodic functions, then $p_j \searrow p_0$, $q_j \nearrow q_0$, for some 2π-periodic functions p_0 and q_0. Applying Theorem 4.7 we have

$$\int_0^{2\pi} (p_0(x) - q_0(x)) e^{ikx} dx = \lim_{j \to \infty} \int_0^{2\pi} (p_j - q_j)(x) e^{ikx} dx =$$

$$= \lim_{j \to \infty} a_k \frac{\hat{g}_j(k) - \hat{f}_j(k)}{\hat{g}(k)} = 0$$

by (4.10). Hence $p_0 = q_0$. Therefore $\lim (h * \mu(x) - q(x)) = 0$. The last statement also follows readily. Q.E.D.

Let $\tilde{W}(\mathbf{R}^+)$ be the space of all locally Riemann integrable functions on $\mathbf{R}^+$ such that

$$\sum_{k=-\infty}^{\infty} \|s g(x)\|_{L^\infty[2^k, 2^{k+1})} < \infty.$$

By a change of variables from Theorem 4.8, we have the following result.

Corollary 4.9. *Let F be a non-negative Borel measurable function on* $\mathbf{R}^+$ *such that* $\limsup_{k \to \infty} \int_{2^k}^{2^{k+1}} F(s)\, ds/s < \infty$. *Suppose* $0 \leqslant g_1 \in \tilde{W}(\mathbf{R}^+)$ *is such that* $\int_0^\infty g_1(s) s^{i\xi} ds \neq 0$ *for all* $\xi \in$, $\mathbf{R}$, *and suppose*

$$\lim_{T\to\infty}\left[\int_0^\infty F(Ts)\,g_1(s)\,ds - P(T)\right] = 0$$

for some bounded multiplicative periodic function p of period λ *(i.e.* $P(\lambda T) = P(T)$*). Then for any* $g_2 \in \tilde{W}(\mathbf{R}^+)$*, we have*

$$\lim_{T\to\infty}\left[\int_0^\infty F(Ts)\,g_2(s)\,ds - Q(T)\right] = 0$$

for some Q *of the same type. Moreover if* $P(x) = \sum_{k=-\infty}^{\infty} a_k s^{2\pi i k/\lambda}$, $Q(s) = \sum_{k=-\infty}^{\infty} b_k s^{2\pi i k/\lambda}$*, then*

$$b_k = a_k(\hat{g}_1(k)/\hat{g}_2(k)), \quad k \in \mathbf{N}$$

where $\hat{g}(k) = \int_0^\infty g(s)\,s^{2\pi i k/\lambda}\,ds$.

The following result is the main application of Theorem 4.8.

Theorem 4.10. *For* $0 \leqslant \alpha < n$, $f \in B_\alpha^2(\mathbf{R}^n)$*, the following two expressions are equivalent*:

$$\lim_{t\to 0}\left[\frac{1}{t^{n+\alpha}}\int_{\mathbf{R}^n} |W_t(f)(x)|^2\,dx - P(t)\right] = 0; \tag{4.11}$$

$$\lim_{T\to\infty}\left[\frac{1}{T^{n-\alpha}}\int_{|y|\leqslant T} |f(y)|^2\,dy - Q(T)\right] = 0, \tag{4.12}$$

for some P, Q bounded multiplicative periodic functions with period λ*. The functions P, Q are related by*

$$b_k = a_k\cdot\frac{\hat{\psi}(k)}{\hat{\phi}(k)}, \quad k \in \mathbf{N},$$

where $P(1/s) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k/\lambda}$, $Q(s) = \sum_{k=-\infty}^{\infty} b_k e^{2\pi i k/\lambda}$, $\phi(s) = s^{-\alpha-1} J_{n/2}^2(2\pi s)$ *and* $\psi(s) = s^{n-\alpha-1}\chi_{[0,1]}(s)$.

Proof. Let $\tilde{P}(s) = P(1/s)$. By (4.3) and (4.2), we can rewrite (4.11) and (4.12) as:

$$\lim_{T\to\infty}\left[\int_0^\infty F(Ts)\,\phi(s)\,ds - \tilde{P}(T)\right] = 0;$$

$$\lim_{T\to\infty}\left[\int_0^\infty F(Ts)\,\psi(s)\,ds - Q(T)\right] = 0.$$

The theorem follows from the Corollary 4.9 by observing the following:

(1) Since $F(s) = s^{\alpha}\int_{S_{n-1}}|f(su)|^2\,du$ and $f \in B_\alpha^2(\mathbf{R}^n)$ we have

$$\int_{2^k}^{2^{k+1}} F(r)\frac{1}{r}\,dr = \int_{2^k}^{2^{k+1}}\left[\int_{S_{n-1}}|f(ru)|^2\,du\right]r^{\alpha-1}\,dr \leqslant$$

$$\leqslant \left(\frac{1}{2^{k+1}}\right)^{n-\alpha}\int_{2^k \leqslant |x| \leqslant 2^{k+1}}|f(x)|^2\,dx \leqslant$$

$$\leqslant \sup_{t \geqslant 1}\frac{1}{T^{n-\alpha}}\int_{|x| \leqslant T}|f(x)|^2\,dx < \infty .$$

(2) $\psi, \phi \in \tilde{W}(\mathbf{R}^+)$ (since $\phi(s) = O(s^{n-\alpha-1})$ as $s \to 0$, and $= O(s^{-(\alpha+2)})$ as $s \to \infty$).

(3) $\int_0^\infty \phi(s)s^{i\xi}\,ds \neq 0$ for all $\xi \in \mathbf{R}$: this follows from substituting $u = v = \frac{n}{2}$, $a = 2\pi$, $\eta = \alpha + 1 - i\xi$ into the identity for $\int_0^\infty J_u(at)J_v(at)s^{-\eta}\,ds$ in ([**Wa**], p. 403), such that for $\xi \in \mathbf{R}$,

$$\int_0^\infty \phi(s)s^{i\xi}\,ds = \int_0^\infty J_{\frac{n}{2}}^2(2\pi s)\,s^{-(\alpha+1-i\xi)}\,ds =$$

$$= \frac{\pi^{\alpha-i\xi}\,\Gamma(\alpha+1-i\xi)\,\Gamma\big(n-\frac{1}{2}(\alpha-i\xi)\big)}{2\,\Gamma^2\big(\frac{1}{2}(\alpha-i\xi)+1\big)\,\Gamma\big(\frac{1}{2}(\alpha-i\xi)+n+1\big)} \neq 0.$$

(4)

$$\int_0^\infty \psi(s)s^{i\xi}\,ds = \int_0^1 s^{n-\alpha-1+i\xi}\,ds = \frac{1}{n-\alpha+i\xi} \neq 0$$

for any $\xi \in \mathbf{R}$. Q.E.D.

Corollary 4.11. *For* $0 \leqslant \alpha < n$, $f \in B_\alpha^2(\mathbf{R}^n)$,

$$\lim_{T\to\infty}\frac{1}{T^{n-\alpha}}\int_{|x|\leqslant T}|f(x)|^2\,dx = \lim_{t\to 0}\frac{C_\alpha}{t^{n+\alpha}}\int_{\mathbf{R}^n}|W_tf)(y)|^2\,dy,$$

if either one limit exists, where $C_\alpha = \hat{\psi}(0)/\hat{\phi}(0)$, *with* $\hat{\phi}(0) = \int_0^\infty s^{-\alpha-1}J_{n/2}^2(2\pi s)\,ds$ *and* $\hat{\psi}(0) = \int_0^1 s^{n-\alpha-1}\,ds = (n-\alpha)^{-1}$.

Corollary 4.12. *Let μ be a bounded Borel measure, then Theorem 4.10 and Corollary 4.11 also hold if we replace f by $\hat{\mu}$, and $W_t(f)(x)$ by $\mu(B_t(x))$.*

5. Fourier Asymptotics

In this section we will make use of the Tauberian results in the last section to improve several results of STRICHARTZ [S3].

Theorem 5.1. *Let $\{S_j\}_{j=1}^m$ be contractive similarities satisfying the open set condition with respect to U. Suppose that μ is the self-similar measure with positive weights $\{a_j\}_{j=1}^m$ such that $\mu(\partial U) = 0$. Then for β satisfying $\sum_{j=1}^m a_j^2 \varrho_j^{-\beta} = 1$, we have*

$$\lim_{T\to\infty}\left[\frac{1}{T^{n-\beta}}\int_{|x|\leqslant T}|\hat{\mu}(x)|^2\,dx - Q(T)\right] = 0,$$

where $Q > 0$. Furthermore:

(i) *If $\{-\ln \varrho_j : j = 1, \ldots, m\}$ is non-arithmetic, then $Q \equiv c$ for some constant c.*

(ii) *Otherwise, let $((\ln\lambda)Z)$, $\lambda > 1$ be the lattice generated by $\{-\ln \varrho_j : j = 1, \ldots, m\}$, then $Q(\lambda T) = Q(T)$.*

Proof. By Theorem 3.7, Corollary 4.12 all the conclusions except that $Q > 0$ hold. For this we let $H(T) = \frac{1}{T^{n-\beta}}\int_{|x|\leqslant T}|\hat{\mu}(x)|^2\,dx$, if Q is not strictly positive then there exists a $T_0 > \lambda$ such that $\lim_{k\to\infty} H(T_0\lambda^k) = Q(T_0) = 0$. Hence for any $1 < T < T_0$

$$H(T) \leqslant \frac{1}{T^{n-\beta}}\int_{|x|\leqslant T_0}|\hat{\mu}(x)|^2\,dx \leqslant \frac{T_0^{n-\beta}}{T^{n-\beta}}H(T_0).$$

Hence $H(T\lambda^k) \leqslant \frac{T_0^{n-\beta}}{T^{n-\beta}} H(T_0\lambda^k)$ for all $k \geqslant 0$. This will force $Q = 0$ and imply $\lim_{t\to 0} V_\beta(t;\mu) = 0$ (Corollary 4.5). It contradicts Theorem 3.5 (or Theorem 3.7). Q.E.D.

We remark that the above theorem extends ([S3], Corollary 5.3), where the contractive similarities must satisfy the strong open set condition, and the above limit is proved in a weaker sense.

For any positive Borel measure μ on $\mathbf{R}^n$ and for any Borel measurable function f, we use μ_f to denote the measure $\mu_f(E) = \int_E f\,d\mu$ for any Borel subset E. In [S1], STRICHARTZ observed that several classical theorems can be summaried as

$$\lim_{T\to\infty} \frac{1}{T^{n-\alpha}} \int_{|x|\leqslant T} |(\mu_f\hat{)}(x)|^2\,dx = \int_{\mathbf{R}^n} |f|^2\,d\mu, \qquad f\in L^2(\mu),$$

where μ is the α-Hausdorff measure and $\alpha = 0, 1, \ldots, n$. This identity cannot be extended to non-integral α in general, the reader can refer to **[S1]** and **[L1]** for various inequality analogs. If μ is a self-similar measure with natural weights (i.e. $a_j = \varrho_j^\beta$ where $\sum_{j=1}^m \varrho_j^\beta = 1$) and $\{S_j\}_{j=1}^m$ satisfies the strong open set condition, STRICHARTZ (**[S3]**, Corollary 5.3) has some success in extending the above identity. In the following we will sharpen his result with a much simpler proof. Recall the μ is locally uniformly α-dimensional if $\mu(B_r(x)) \leqslant c\,r^\alpha$ for any ball $B_r(x)$ in $\mathbf{R}^n$ and $0 < r < 1$. If μ is a self-similar measure with natural weights, then μ is locally uniformly α-dimensional.

Lemma 5.2. *Let μ be a locally uniformly α-dimensional positive measure, then there exist c_1 and c_2 such that for any $f\in L^2(d\mu)$, the following inequalities*

$$\sup_{0<t<1} \frac{1}{t^{n+\alpha}} \int |(\mu_f)(B_t(x))|^2\,dx \leqslant c_1 \|f\|^2_{L^2(d\mu)},$$

$$\sup_{T>1} \frac{1}{T^{n-\alpha}} \int |(\mu_f\hat{)}(y)|^2\,dy \leqslant c_2 \|f\|^2_{L^2(d\mu)}$$

hold.

Proof. The first inequality can be obtained using ([**L1**], Corollary 2.4). It also follows from the following simple proof: By Schwartz Inequality and a change of variables, we have

$$\frac{1}{t^{n+\alpha}} \int |\mu_f(B_t(x)|^2\,dx = \frac{1}{t^{n+\alpha}} \int \left(\int_{B_t(x)} f(\xi)\,d\mu(\xi)\right)^2 dx \leqslant$$

$$\leqslant \frac{1}{t^{n+\alpha}} \int \left(\mu(B_t(x))\cdot \int_{B_t(x)} |f(\xi)|^2\,d\mu(\xi)\right) dx \leqslant$$

$$\leqslant \frac{c}{t^n} \int\int \chi_{B_t(x)}(\xi)\,|f(\xi)|^2\,d\mu(\xi)\,dx =$$

$$= \frac{c}{t^n} \int \left(\int \chi_{B_t(\xi)}(x)\,dx\right) |f(\xi)|^2\,d\mu(\xi) \leqslant$$

$$\leqslant c_1 \int |f(\xi)|^2\,d\mu(\xi) = c_1 \|f\|^2_{L^2(d\mu)}.$$

To prove the second inequality, we need observe that $(\mu_f)\hat{} \in B_\alpha^2$ (**[L1]**, Theorem 3.5) so that Theorem 4.4 can be applied to the first inequality and yields the result. Q.E.D.

For any $J \in \Lambda$, we define $f_J = \chi_{\overline{U}_J}$, then $\mu_{f_j} = a_J \mu_J$. By a change of variable, we have

$$\frac{1}{t^{n+\beta}} \int \mu_{f_J}^2 (B_t(x))\, dx = a_J^2 \frac{1}{t^{n+\beta}} \int \mu_J^2 (B_t(x))\, dx =$$

$$= a_J^2 \varrho_J^{-\beta} (P(\varrho_J^{-1} t) + o(1)) = \qquad \text{(by Theorem 3.7)}$$

$$= a_J P(t) + o(1) \tag{5.3}$$

as $t \to 0$. Also for any $J \neq J'$ with $|J| = |J'|$, the same argument as in Lemma 3.6 yields

$$\lim_{t\to 0} \int \mu_{f_J}(B_t(x)) \cdot \mu_{f_{J'}}(B_t(x))\, dx$$

$$= a_J a_{J'} \lim_{t\to 0} \int \mu_J(B_t(x)) \cdot \mu_{J'}(B_t(x))\, dx = 0. \tag{5.4}$$

Theorem 5.3. *Suppose that μ is a self-similar measure with natural weights $a_j = \varrho_j^{-\beta}$, and suppose the open set condition holds with the open set U satisfying $\mu(\partial U) = 0$. Then for any $f \in L^2(d\mu)$ we have*

$$\lim_{t\to 0} \left(\frac{1}{t^{n+\beta}} \int |(\mu_f(B_t(x))|^2 dx - P(t) \int |f|^2 d\mu \right) = 0; \tag{5.5}$$

$$\lim_{T\to\infty} \left(\frac{1}{T^{n-\beta}} \int_{|y| \leqslant T} |(\mu_f)\hat{}(y)|^2 dy - Q(T) \int |f|^2 d\mu \right) = 0, \tag{5.6}$$

where P and Q are defined in Theorem 3.7 and Theorem 5.1 respectively.
In particular, if $\{\varrho_j\}$ is non-arithmetic, then there exist $c_1, c_2 > 0$ with

$$\lim_{T\to\infty} \frac{c_1}{T^{n-\beta}} \int_{|x| \leqslant T} |(\mu_f)\hat{}(x)|^2 dx = \lim_{t\to 0} \frac{c_2}{t^{n+\beta}} \int_{\mathbf{R}^n} |\mu_f(B_t(y))|^2 dy = \int |f|^2 d\mu$$

for any $f \in L^2(\mu)$.

Proof. For any $f \in L^2(d\mu)$, we may approximate f in $L^2(d\mu)$ by functions of the form $\sum_{|J|=N} c_J f_J$. By (5.3) and (5.4) it is easy to see that (5.5) is true for such functions. Hence we may obtain (5.5) for any $f \in L^2(d\mu)$ throught a routine limiting argument because of the first inequality of Lemma 5.2. Again using the Tauberian Theorems in Section 4 we can obtain (5.6) from (5.5). Q.E.D.

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KA-SING LAU and JIANRONG WANG
Department of Mathematics and Statistics
University of Pittsburgh
Pittsburgh, PA 15260, USA